Solutions to a Couple of IMO-2019 Problems

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I. Introduction & Background

IMO 2019 is ongoing in Bath, United Kingdom. Yesterday (July 16, 2019) was said to be the first day. Some friends posted the three problems of day 1 online, and I took the liberty of working on the first two problems. I did not try the 3rd problem due to lack of interest as well as concern of time I might have to spend.

The following two sections are my solutions to the first two problems. This represents no significance. It is purely for fun remembering old high school days.

II. Problem One

A. Problem

Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)). \tag{1}$$

B. Solution

In (1), let $b = 0, a = x \in \mathbb{Z}$, we get

$$f(2x) + 2f(0) = f(f(x)), \tag{2}$$

and let $a = 0, b = x \in \mathbb{Z}$, we get

$$f(0) + 2f(x) = f(f(x)). \tag{3}$$

Comparing (2) and (3), we get

$$f(2x) = 2f(x) - f(0). \tag{4}$$

Applying (4) and (3) in (1), we get

$$2f(a) + 2f(b) - f(0) = f(0) + 2f(a + b),$$

i.e.

$$f(a + b) = f(a) + f(b) - f(0). \tag{5}$$

In (5), let $a = x \in \mathbb{Z}, b = 1$. Then

$$f(x + 1) - f(x) = f(1) - f(0). \tag{6}$$

Hence

$$f(x) = xd + f(0) \quad \text{for some } d \in \mathbb{Z}. \tag{7}$$

Applying (7) and (3) leads to

$$2dx + 3f(0) = d^2x + (d + 1)f(0).$$

Hence

$$\begin{cases} d^2 = 2d \\ (d - 2)f(0) = 0. \end{cases} \tag{8}$$

From (8), if $d = 0$, then $f(0) = 0$ and $f(x) \equiv 0$; Otherwise, d=2. Hence the final solutions are:

$$f(x) \equiv 0$$

or

$$f(x) = 2x + m \text{ for any } m \in \mathbb{Z}.$$

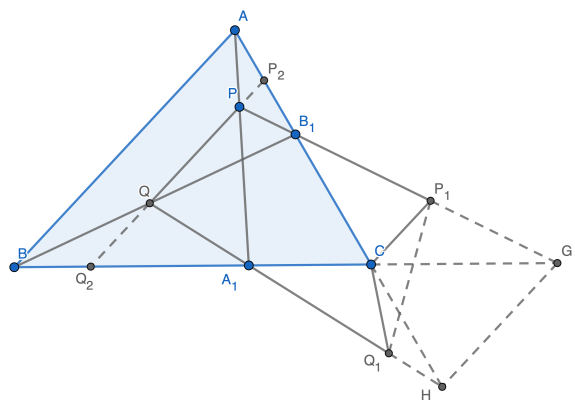
III. Problem Two

A. Problem

In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Fig. 1: Geometric illustration with auxiliary line segments



B. Proof

Extend PQ to intersect AC at P_2, and to intersect BC at Q_2. Extend PP_1 to intersect BC at G, and extend QQ_1 to intersect AC at H. Connect line segments as being shown in Figure 1.

We intend to first prove that **GH is parallel to PQ and AB**. For this, we repeatedly apply *Menelaus's Theorem* on different configurations.

Specifically, applying the theorem on ΔP_2CQ_2 and line segment PG leads to

$$\frac{GQ_2}{GC} \times \frac{B_1C}{B_1P_2} \times \frac{PP_2}{PQ_2} = 1 \Rightarrow \frac{GQ_2}{GC} = \frac{B_1P_2}{B_1C} \times \frac{PQ_2}{PP_2}, \tag{9}$$

and applying the theorem on ΔP_2CQ_2 and line segment QH leads to

$$\frac{HP_2}{HC} \times \frac{A_1C}{A_1Q_2} \times \frac{QQ_2}{QP_2} = 1 \Rightarrow \frac{HP_2}{HC} = \frac{A_1Q_2}{A_1C} \times \frac{QP_2}{QQ_2}. \tag{10}$$

Dividing (10) by (9) leads to

$$\frac{HP_2}{HC} / \frac{GQ_2}{GC} = \frac{A_1Q_2}{A_1C} \times \frac{PP_2}{PQ_2} \times \frac{B_1C}{B_1P_2} \times \frac{QP_2}{QQ_2}. \tag{11}$$

Applying the theorem on ΔP_2CQ_2 and line segment AA_1 and BB_1 respectively leads to

$$\frac{PP_2}{PQ_2} \times \frac{A_1Q_2}{A_1C} \times \frac{AC}{AP_2} = 1 \Rightarrow \frac{PP_2}{PQ_2} \times \frac{A_1Q_2}{A_1C} = \frac{AP_2}{AC}, \tag{12}$$

$$\frac{B_1C}{B_1P_2} \times \frac{QP_2}{QQ_2} \times \frac{BQ_2}{BC} = 1 \Rightarrow \frac{B_1C}{B_1P_2} \times \frac{QP_2}{QQ_2} = \frac{BC}{BQ_2}. \tag{13}$$

Comparing (11), (12) by (13) leads to

$$\frac{HP_2}{HC} / \frac{GQ_2}{GC} = \frac{AP_2}{AC} / \frac{BQ_2}{BC} = 1. \tag{14}$$

The last equality was due to the fact that PQ and P_2Q_2 are parallel to AB. Hence we complete the proof of the statement that **GH is parallel to AB and PQ**. We are now ready to show that the conclusion of the problem holds.

Since GH is parallel to AB, we have $\angle CGH = \angle ABC = \angle CQ_1P_1$. Hence points C, G, H, Q_1 are co-cyclic.

Similarly, $\angle CHG = \angle BAC = \angle CP_1B_1$. Hence points C, P_1, G, H are co-cyclic.

Note that the two co-cyclic points groups share three common points C, G, H. Hence the five points C, P_1, G, H, Q_1 are all co-cyclic. Hence

$$\angle CGP_1 = \angle CQ_1P_1.$$

Finally,

$$\angle P_2PP_1 = \angle PQ_2G + \angle CGP_1 = \angle ABC + \angle CQ_1P_1 = \angle CQ_1A_1 + \angle CQ_1P_1 = \angle QQ_1P_1.$$

Hence points P, P_1, Q_1, Q are co-cyclic.