A Solution to Problem 6 of IMO 2019

Xiaohai Zhang July 18, 2019

I. Introduction & Background

IMO 2019 just ended in Bath, United Kingdom a couple of days ago. Problem 6 is a geometry problem. It is supposed to be the hardest one as it is the last. I gave it a bit thinking, figured out the general ideas of proving but did not follow through. I then noticed a discussion page at:

https://artofproblemsolving.com/community/c6h1876745_geometry_finale_incircles_and_concurrency.

It is a problem that can take advantage of advanced knowledges about projective geometry. Some proofs in the discussion page are ingenious but porous and hard to follow. Here I intend to give a proof that is relatively verbose but easier to follow. I make efforts to ensure every claim is properly substantiated with common knowledges.

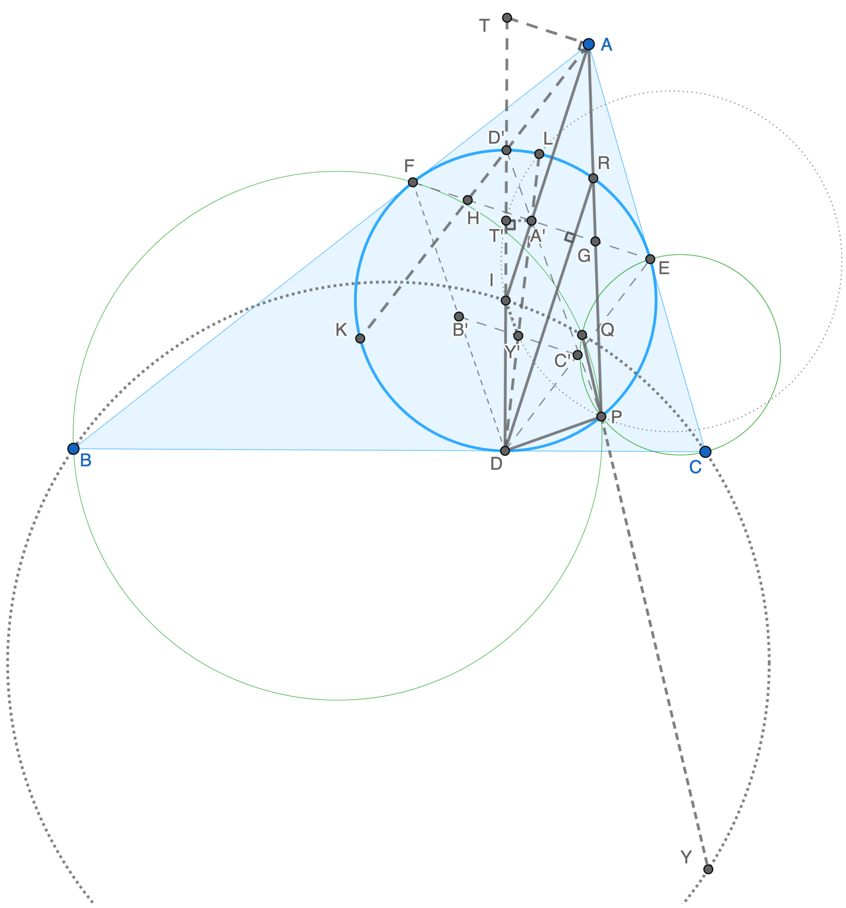
Lastly, I would like to express my thanks to Anant Mudgal of India for proposing such an excellent problem (so I read).

II. Problem Statement

Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

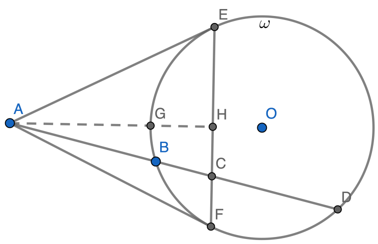
III. Proof



A. Lemmas

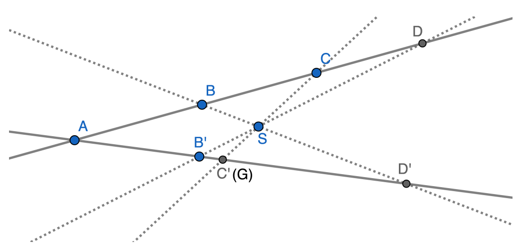
We first layout a couple of lemmas required for solving this problem.

Lemma 1. *Let ω be a circle, A be a point outside ω, and E and F be two tangent points from A to ω. Let a line between AE and AF intersect ω, EF and ω again at points B, C, D respectively. Then $(A, C; B, D) = -1$ is a harmonic quartet.*



Proof. $AF^2 = AB \times AD = AB^2 + AB \times (CD + BC) = (AC - BC)^2 + (AC - BC) \times (CD + BC) = AC^2 + BC^2 - 2AC \times BC + AC \times CD + AC \times BC - BC \times CD - BC^2 = AC^2 - AC \times BC + AC \times CD - BC \times CD = AC^2 - AD \times BC + AC \times CD = AC^2 - AD \times BC + (AB + BC) \times CD = AC^2 + BC \times CD - AD \times BC + AB \times CD = AC^2 + FC \times CE - AD \times BC + AB \times CD = AC^2 + (FH - HC) \times (FH + HC) - AD \times BC + AB \times CD = AC^2 - HC^2 + FH^2 - AD \times BC + AB \times CD = AH^2 + FH^2 - AD \times BC + AB \times CD = AF^2 - AD \times BC + AB \times CD$. Hence $AD \times BC = AB \times CD$, or $\frac{AB}{CB} : \frac{AD}{CD} = -1$ when signs are included. □

Lemma 2. *Let $(A, C; B, D) = -1$ and $(A, C^{'}; B^{'}, D^{'}) = -1$ be two distinct harmonic quartets with a common point A. Then $CC^{'}, BD^{'}, BD^{'}$ are concurrent.*



Proof. Let $BD^{'}, B^{'}D$ intersect at point S, and let CS intersect line $\overline{AB^{'}}$ at G. Then $B^{'}, D', G$ are images of D, B, C respectively from the perspective of point S. By the fact that cross ratio is invariant under projection and $(A, C; B, D) = -1$, we have $(A, G; D^{'}, B^{'}) = -1$. Comparing with $(A, C^{'}; B^{'}, D^{'}) = -1$, we know $C^{'}$ and G colocates. □

B. Proof of the problem

Now we formally begin the proof of the problem. To improve readability, we divide the proof into multiple claims. We will be dealing with inverse points relative to the incircle ω at times. We use the convention that the inverse point of a point X is denoted by $X^{'}$ (with one exception of $D^{'}$ which is a mere antipode).

Let the line through point A perpendicular to AI meets line $\overline{DI}$ at T. It suffice to show that T, Q, P are collinear. Let $D^{'}$ be D's antipode, $G = AP \cap EF, H = \overline{AD^{'}} \cap EF$, and K be the other meet point between $\overline{AD^{'}}$ and incircle ω. Since both AI and DR are perpendicular to EF, AI bisects $\widehat{D^{'}R}$. AK and AP are symmetric around AI.

Claim 1. *Points $A^{'}, B^{'}, C^{'}$ locate at the middle of line segments EF, DF, DE respectively.*

Proof. Recall that $A^{'}, B^{'}, C^{'}$ denote inverse points of A, B, C. Given the fact that ω is an incircle, the conclusion is basic and obvious. □ □

Claim 2. *Perpendicular foot of $A^{'}$ onto line DI is the inverse point $T^{'}$ of T.*

Proof. Note both $\angle A^{'}T^{'}T$ and $\angle A^{'}AT$ are right angles. Hence points $A^{'}, T^{'}, T, A$ are concyclic. $IT^{'} \times IT = IA^{'} \times IA = r^2$ where r is the radius of circle ω (recall that $A^{'}$ is the inverse point of A.). □

Claim 3. *Points B, I, Q, C are concyclic.*

Proof. $\angle BQC = \angle BQP + \angle CQP = \angle BFP + \angle CEP = \frac{1}{2}(\widehat{PDKF} + \widehat{PE}) = 180° - \frac{1}{2}(\angle ABC + \angle ACB) = \angle BIC$. □

Claim 4. *Quadrilateral $BICY$ and $FREP$ are similar and harmonic where Y is the other intersection between line $\overline{QP}$ and the circumcircle of $BIQC$.*

Proof. By the construction method, quadrilateral $FREP$ is harmonic. Note the symmetry between $D^{'}$ and R, $\widehat{FR} = \widehat{D^{'}E} = 180° - \widehat{EPD} = \angle C = \widehat{BI}$. Similarly, $\widehat{RE} = \widehat{FD^{'}} = 180° - \widehat{FKD} = \angle B = \widehat{IC}$. $\widehat{FP} = 2\angle BFP = 2\angle BQP = \widehat{BY}$. Hence Quadrilateral $BICY$ and $FREP$ are similar. And the similarity leads to $BICY$ being harmonic as well. ☐

Claim 5. *The inverse point $Y^{'}$ of Y is the midpoint of $B^{'}$ and $C^{'}$.*

Proof. Harmonic property is invariant under the inverse operation. Given that the quadrilateral $BICY$ is harmonic, its image is harmonic, i.e $(B^{'}, C^{'}; \infty, Y^{'}) = -1$. Hence $B^{'}Y^{'} = Y^{'}C^{'}$, and $Y^{'}$ is the midpoint of $B^{'}$ and $C^{'}$. ☐

Claim 6. *Points $Y^{'}, I, T^{'}, L$ are concyclic where $L = \overline{DA^{'}} \cap \omega$.*

Proof. Note $\angle DLD^{'}$ and $\angle A^{'}T^{'}D^{'}$ are both right angles. Hence points $T^{'}, A^{'}, L, D^{'}$ are concyclic, and $DT^{'} \times DD^{'} = DA^{'} \times DL$. Note I and $Y^{'}$ are midpoints of $DD^{'}$ and $DA^{'}$ respectively. We have $DT^{'} \times DI^{'} = DY^{'} \times DL$. Hence $Y^{'}, I, T^{'}, L$ are concyclic. ☐

Claim 7. *Points $D^{'}, A^{'}, P$ are collinear.*

Proof. From Lemma 1, $ARGP$ and $AD^{'}HK$ are both harmonic quartets. From Lemma 2, $D^{'}P$, RK and GH are concurrent. Symmetry of the two quartets around AI requires the intersection point being $A^{'}$, the midpoint of EF. ☐

Claim 8. *Points $P, Y^{'}, I, L$ are concyclic.*

Proof. Note I and $Y^{'}$ are midpoints of $DD^{'}$ and $DA^{'}$ respectively. $IY^{'} \parallel D^{'}A^{'}$, $IY^{'} \parallel D^{'}P$ and $\angle PIY^{'} = \angle IPD^{'} = \angle ID^{'}P = \angle PD^{'}D = \angle PLD = \angle PLY^{'}$. Hence points $P, Y^{'}, I, L$ are concyclic. ☐

From Claim 6 and Claim 8, we have points $P, Y^{'}, I, T^{'}, L$ are concyclic. By inversion property, we know points P, Y, T are collinear. Hence P, Q, T are collinear. The proof is complete.